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A stochastic programming model to perform a production smoohing function

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A STOCHASTIC PROGRAMMING MODEL

TO PERFORM A PRODUCTION SMOOTHING FUNCTION

by

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ABSTRACT

If statistical variation is introduced into at least one of the parameters of a linear programming problem, it becomes one of stochastic programming. Scheduling production resources to optimally satisfy a random consumer demand is a problem of this type. A stochastic programming model to perform a production smoothing function for the case of a normally distributed consumer demand with known mean and variance is presented in this paper. The formulation results in a programming problem to minimize a convex-separable objective function subject to a set of linear constraints. The convex objective is approximated over the range of interest by a piecewise-linear function for solution by the simplex algorithm.

I - INTRODUCTION

I-A. Purpose and Scope

A class of problems which frequently arises in industrial or economic situations is that of maximizing or minimizing a function of any number of variables subject to a set of restrictions. Often problems of this type involve a large number of variables and complex interrelationships among the variables. Although such problems can conceptually be solved by classical methods such as the use of Lagrangian multipliers, computation is very often impractical or impossible.

Since 1947 numerous iterative methods have been developed to solve certain subclasses of optimization problems which deal with many variables and a large number of restraining conditions. These are the techniques of mathematical programming. Such problems arise when there are a number of activities to be performed and there are alternative ways of doing them, and resources or facilities are not available for performing each activity in the most effective way [7].

The most powerful and most fully developed methods of mathematical programming are those which deal with the optimization of a linear function subject to linear restrictions. The linear programming problem can be classified according to the characteristics of the parameters of the objective function and of the constraints. If the parameters are known and fixed the problem is one of deterministic linear programming. If the coefficients are allowed to vary the problem becomes one of parametric programming. A third type, stochastic programming, refers to that class of problems in which statistical variation in any or all of the problem parameters is considered.

There are two broad classifications of stochastic programming problems: (1) Those in which the parameters have known probability distribution functions, and (2) those in which the distribution functions are not known. The first is said to involve risk while the latter involves uncertainty [1]. Most of the modern literature in the area of stochastic programming is concerned with those problems in which the form and the parameters of the distributions are known and constant in time. The problem to be presented here is also one which, according to the definitions above, is concerned with a situation involving risk.

The purpose of this paper is to formulate a model which will perform a production smoothing function for a single product production shop when demand is not known deterministically. The problem is one of determining an optimal policy in which production decisions must be made for a finite number of time periods in the future before the actual demand becomes known. In this application of stochastic linear programming the random variable, consumer demand for a product, appears both in the objective equation and in the constraints of the problem. The study will treat consumer demand as a normally distributed random variable with known mean and variance.

I-B. Background

Since stochastic programming is a logical extension of the linear programming problem, the subject is perhaps best introduced by a discussion of the linear problem in which statistical variation is not considered. The maximization problem may be stated as follows:

$$\text{Maximize} \quad F = \sum_{j=1}^{n} c_j x_j$$

Subject to the constraints:

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \qquad i = 1, \ldots, m$$

$$x_j \geq 0 \qquad j = 1, \ldots, n$$

In a problem of this form the c_j may be contributions to profit for say a manufacturing operation, the a_{ij} technological constants of the operations, the b_i capacity limitations and the x_j the unknown quantities of the various products to manufacture. The solution to the problem is a set of positive values of x_j which maximizes F while satisfying the inequalities.

The solution can be obtained using the simplex technique. This algorithm requires that the constraints (hyperplanes in n dimensions) intersect in a convex set. The polyhedron formed by the intersecting hyperplanes contains the set of all feasible solutions; i.e., solutions which simultaneously satisfy the restraining inequalities. Since the solution space has a finite number of vertices (extremum) and the optimal solution occurs at a vertex of the solution space, the optimum can be obtained in a finite number of iterations. A proof for this is available in any textbook on linear programming; e.g., see Gass [14].

The algorithm evaluates adjacent extreme points with non-decreasing (for maximization problems) values of the objective function. When a value for the function is obtained which is less than the value of the previous iteration the algorithm is terminated. The requirement that the constraint set be convex assures that a global rather than a local minimum or maximum has been obtained.

For notational convenience is the discussion which follows the linear programming problem can be expressed in matrix form.

Maximize	$F = C^T X$
Subject to:	$AX \leq b$
Where:	C is an n x 1 vector - cost vector
	A is an m x n matrix - technology matrix
	b is an m x 1 vector - requirements vector
	X is an n x 1 vector - activity vector

The linear program becomes stochastic if randomness is introduced into the A matrix or the b and c vectors. A question that arises in extending linear programming to include randomness is how to handle the notion of feasibility. It is to answering this question that the literature in the area of stochastic programming is directed.

One of the most obvious methods which can be used to obtain a solution to the stochastic problem is to simply replace the random elements by their expected values. This technique is computationally advantageous in that the linear format is retained and the problem is not enlarged. The disadvantages are that infeasibility results fifty-per cent of the time, and the resultant solution is not necessarily the most optimal. The expected value solution will, however, provide a good starting solution on which to base further improvement.

Another method of resolving the problem of randomness is the technique which Madansky [18] calls the "fat" formulation. This method restricts the choice of values of the activity vector X such that X is a feasible solution regardless of what values of A and b are subsequently observed. That is, the values of X must be taken from the set of solutions which are "permanently feasible". A choice of the activity vector is then made from the set of permanently feasible solutions which will optimize the objective function. One difficulty in this formulation is that the permanently feasible set may be empty; i.e., there is no solution which is feasible for all A and b. This is likely when the distribution of A or b is defined over the entire real line. Of course, the chief objection to the "fat" solution is that it is extremely pessimistic. The solution, while optimal for values of X in the permanently feasible set, may certainly not be optimal for the most probable values of A and b.

In most practical situations, infeasibility does not imply complete system failure. For example, in a production system a restriction of the model might be that consumer demand must be satisfied.

If it were found that after the actual demand materialized shortages were incurred the solution provided by the model would be interpreted as being infeasible. However, the additional product required to meet demand could either be backordered or purchased from another manufacturer. A premium cost is usually paid, but the "infeasible" solution does not imply system failure. Of course there are other situations where infeasibility cannot be tolerated in which case the "fat" formulation provides the best solution.

A more realistic approach which partially overcomes the objections to the pessimistic solution is the so called "slack" formulation. This technique results in a two-stage problem [18]. The rationale of the two-stage case is as follows: A choice of the activity vector X is made, and the values of random matrix A and random vector b are later observed. If, in light of the observed A and b, X is infeasible an adjustment is made in the second stage to compensate for any discrepancies between AX and b, but at a penalty cost. The production system example cited above is of this type.

An algorithm for solving the two-stage in which only the b vector is random can be found in reference [9]. The problem is one of minimizing a convex function of expected costs subject to a set of linear constraints. Dantzig and Ferguson [10] have applied this technique to the problem of scheduling aircraft to meet an uncertain demand.

Another approach to stochastic programming, that of Charnes and Cooper [4,5,6], is to specify a probability with which the problem constraints are to be achieved. This formulation is called "chance-constrained" programming. It reduces to the "fat" formulation if the probability specified for achieving the problem constraints is one.

The chance-constrained problem is of the form:

$$\text{Maximize: } f(c,x)$$

$$\text{Subject to: } PR\,[AX \leq b] \geq \alpha$$

Where α is an m x 1 vector, sometimes called a confidence vector [2], the elements of which are prescribed constants that are probability measures of the extent to which constraint violations are allowed. Consider an element of α, α_i, associated with the constraint

$$PR\,[\sum_{j=1}^{n} a_{ij}\, x_j \leq b_i] \geq \alpha_i \quad 0 \leq \alpha_i \leq 1.$$

This is a double inequality which is interpreted to mean that the ith constraint may be violated but at most $\beta_i = 1 - \alpha_i$ proportion of the time.

Chance-constrained programming was first applied to a case of scheduling the production of heating oil in which the density function of consumer demand for the product was known. The problem is one of scheduling refinery rates in a manner so as to maintain specified minimum and maximum inventory levels and to meet the sales demand at prescribed levels of probability while minimizing total operational costs [6]. A second application is that of determining a charter policy for an oil company which may supplement its available tanker fleet by chartering additional ships on either a long or short term basis. The objective is to minimize expected charting costs for a fixed planning interval, and to satisfy the demand for tankers at a level of probability consistent with company policies [4].

II - A PRODUCTION SMOOTHING MODEL

II-A. Statement of the Problem

The purpose of a production smoothing model is to determine an optimal balance of production resources to satisfy customer demand for a product. Models presented in the literature are numerous [13,17,20], and have been applied to many and varied industrial problems. Since every business has characteristics which are unique, the models employed differ somewhat. Company policies and physical limitations such as available capacity appear as constraining conditions of the problem.

The models used for production smoothing also vary as to degree of refinement. As the model becomes more sophisticated the size of the resultant problem increases rapidly. In most cases a compromise must be made between the number of variables in the problem and cost of computation. It must also be remembered that since many of the cost parameters of the problem must be estimated, no benefit will be gained by imposing conditions on the model which may be overshadowed by the errors of the cost estimates.

S. E. Elmaghraby [13] has presented a production smoothing model to be used to schedule production over a finite planning horizon in a shop producting small electronic components. The planning horizon is the total number of periods over which the production schedule is to be effective. Due to the nature of the product, manufacturing in this shop is on a batch basis and production is to inventory with consumer demand being supplied from this inventory.

The model is formulated as a linear programming problem with costs incurred for carrying inventory, regular time production, overtime production, idle time, and a change in the level of the workforce. The objective is to obtain a production schedule for N periods in the future which will minimize the sum of the above costs while satisfying the problem constraints. These restrictions include available capacity limitations, and the additional condition that cumulative production plus initial inventory must be greater than or equal to the cumulative customer demand.

Forecasts of demand are made for each of the periods in the planning horizon and production is scheduled to meet the forecasted demand with no stockouts allowed. Since the no stockout condition is based on a forecast and forecasts are subject to error, a buffer or safety stock must be carried as protection against shortages.

The purpose of the development which follows is to modify the formulation to handle forecasted demand as a stochastic variable. More specifically to treat the case of demand being a normally distributed random variable with known mean and variance. The problem then becomes one which can be included in the broad class of problems known as stochastic programming [1]. Under the case of normally distributed demand the no stockout condition must be relaxed to meeting demand at a specified (usually high) level of probability. The objective function must also be altered so as to be an expression for expected costs. As will be seen, it is the latter modification that introduces non-linearity into the problem.

II-B. Mathematical Development

Define the following costs:

C_r = cost of regular time production in dollars per hour.

C_o = cost of overtime production in dollars per hour.

C_s = cost of shortage in dollars per hour per period.

C_I = cost of inventory in dollars per hour per period.

C_h = cost of an increase of workforce in dollars per man.

C_f = cost of a decrease of workforce in dollars per man.

C_e = cost of idle time in dollars per hour.

Also define the following production variables:

X_n = number of hours of regular time production in period n

Y_n = number of hours of overtime production in period n

W_n = the size of the workforce in period n

K_n = number of regular time working hours available in period n

K_n' = number of overtime working hours available in period n

$n = 1,2,\ldots,N$

N = the number of periods (months) in the planning horizon

The restrictions placed upon the planned program are as follows:

1. Regular time production in the nth period cannot exceed the total available capacity. Since K_nW_n is the number of hours available for regular time production during month n,

$$X_n \leq K_nW_n$$

2. Similarly overtime production may not exceed the total overtime capacity. Therefore:

(2) $$Y_n \leq K'_n W_n$$

3. Demand must be met in period n at a specified level of probability, α_n. If I_n represents the number of hours in inventory at the end of period n, this restriction may be stated as:

(3) $$PR\left[I_n \geq 0\right] \geq \alpha_n \qquad n = 1,2,\ldots,N$$

The inventory at the end of period n is the inventory at the end of the previous period, plus the production p_n during n, less the amount shipped during the nth period. That is,

$$I_n = I_{n-1} + p_n - d_n$$

where $p_n = X_n + Y_n$

and d_n = demand in period n in production hours

Expressing I_n as a function of the p_n and d_n

(4) $$I_n = I_o + \sum_{i=1}^{n} p_i - \sum_{i=1}^{n} d_i$$

Substituting (4) into the double inequality (3),

$$PR\left[I_o + \sum_{i=1}^{n} p_i - \sum_{i=1}^{n} d_i \geq 0\right] \geq \alpha_n$$

(5) or, $$PR\left[I_o + \sum_{i=1}^{n} p_i \geq \sum_{i=1}^{n} d_i\right] \geq \alpha_n \qquad n = 1,2,\ldots,N$$

To simplify notation let:

$$P_n = I_o + \sum_{i=1}^{n} p_i$$

and, $$D_n = \sum_{i=1}^{n} d_i$$

That is, P_n and D_n denote cumulative production and demand respectively.

Rewriting (5), $PR\,[P_n \geq D_n] \geq \alpha_n \quad n = 1,2,\ldots,N$.

Assume that the marginal density function $f_n\,(D_n)$ is continuous, and define $\hat{D}_n$ such that:

$$\int_{-\infty}^{\hat{D}_n} f_n\,(D_n)\,d\,D_n = \alpha_n$$

Thus $PR\,[P_n \geq D_n] \geq \alpha_n$

Implies $P_n \geq \hat{D}_n \qquad n = 1,2,\ldots,N$

Summarizing the constraining inequalities:

$$(6) \qquad \begin{aligned} X_n &\leq K_n W_n \\ Y_n &\leq K_n' W_n \qquad n = 1,2,\ldots,N \\ P_n &\geq \hat{D}_n \end{aligned}$$

In order to reflect the cost of changing the level of the workforce form the following equation:

$$\Delta W_n = W_n - W_{n-1} = U_n - V_n$$

Where U_n is the number of employees added and V_n represents the decrease in workforce in period n.

Thus $\qquad W_1 = W_o + U_1 - V_1$

or in general $\quad W_n = W_o + \sum_{i=1}^{n} U_i + \sum_{i=1}^{n} V_i$

The inequalities (6) can now be written in terms of the variables U_n and V_n:

$$X_n \le K_n \left[W_o + \sum_{i=1}^{n} (U_i - V_i)\right]$$

$$\text{(7)} \qquad Y_n \le K_n' \left[W_o + \sum_{i=1}^{n} (U_i - V_i)\right] \qquad n = 1,2,\ldots,N$$

$$P_n \ge \hat{D}_n$$

where W_o is the initial workforce.

The constraint set having been formulated we now proceed to develop an expression for the cost of the production program in terms of the variables of the constraints.

Let I_n = number of hours of production in inventory

S_n = number of hours of production shortage

Denoting the cost of the production program as C_n, the cost for the nth month can be expressed as follows:

$$\text{(8)} \quad C_n = C_r X_n + C_o Y_n + C_h U_n + C_f V_n + C_I I_n + C_s S_n$$

The inventory I_n and the shortage S_n are a function of the random variable D_n which is not known in advance of the program planning. In this case we may choose to optimize the expected value of C_n.

$$E(C_n) = E\left[C_r X_n + C_o Y_n + C_h U_n + C_f V_n + C_I I_n + C_s S_n\right]$$

E is the expected value operator.

Since X_n, Y_n, U_n, and V_n are not functions of D_n and since the expected value of a sum is equal to the sum of the expected value, the equation above can be written:

$$E(C_n) = C_r X_n + C_o Y_n + C_h U_n + C_f F_n + C_I E[I_n] + C_s E[S_n]$$

Since $f_n(D_n)$ is continuous the last two terms can be expressed in terms of the marginal density function $f_n(D_n)$.

$$E[I_n] = \int_{-\infty}^{P_n} (P_n - D_n)\ f\ (D_n)\ d\ D_n$$

And,

$$E[S_n] = \int_{P_n}^{\infty} (D_n - P_n)\ f\ (D_n)\ d\ D_n$$

Recall that the programming planning is to take place over N periods. Thus, the objective is to minimize:

$$(9) \quad \sum_{n=1}^{N} E(C_n) = C_r \sum_{n=1}^{N} X_n + C_o \sum_{n=1}^{N} Y_n + C_n \sum_{n=1}^{N} U_n + C_f \sum_{n=1}^{N} V_n$$

$$+ \sum_{n=1}^{N} [C_I \int_{-\infty}^{P_n} (P_n - D_n)\ f\ (D_n)\ d\ D_n + C_s \int_{P_n}^{\infty} (D_n - P_n)\ f\ (D_n)\ d\ D_n]$$

subject to the constraint set (7).

While the constraint set for the stochastic model remains linear, the objective function does not. Hence, the simplex algorithm for linear programming cannot be applied. However, the objective function will be shown to be convex and can be handled by the methods of convex programming.

II-C. The Normalized Objective Function

In the preceding section it was shown that the objective function was composed of a sum of linear terms and the summation of integrals of the form:

(10) $$g(P_n) = C_I \int_{-\infty}^{P_n} (P_n - D_n)\, f_n(D_n)\, dD_n + C_s \int_{P_n}^{\infty} (D_n - P_n)\, f_n(D_n) dD_n$$

Where:

$$f(D_n) = N(D_n;\, \mu_n,\, \sigma_n) = \frac{1}{\sqrt{2\pi}\,\sigma_n} e^{-\frac{1}{2}\left(\frac{D_n - \mu_n}{\sigma_n}\right)^2}$$

Let:

$$g(P_n) = h(P_n) + k(P_n)$$

$$h = \frac{C_I}{\sqrt{2\pi}} \int_{-\infty}^{P_n} \frac{(P_n - D_n)}{\sigma_n} e^{-\frac{1}{2}\left(\frac{D_n - \mu_n}{\sigma_n}\right)^2} dD_n$$

Define: $t = \dfrac{D_n - \mu_n}{\sigma_n}$ $\qquad D_n = \sigma_n t + \mu_n$

$$\sigma_n dt = dD_n$$

$$h = \frac{C_I}{\sqrt{2\pi}} \int_{-\infty}^{P_n} (P_n - \sigma_n t - \mu_n)\, e^{-\frac{1}{2}t^2} dt$$

$$h = C_I (P_n - \mu_n) \int_{-\infty}^{Z_n} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt + \frac{C_I \sigma_n}{\sqrt{2\pi}} \int_{-\infty}^{Z_n} -te^{-\frac{1}{2}t^2} dt$$

(11) Where $Z_n = \dfrac{P_n - \mu_n}{\sigma_n}$

(12) Let $\Phi(x) = \int_0^x \varphi(t) dt$ and $\varphi(x) = \dfrac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$

$$h = C_I\sigma_n Z_n \left[\frac{1}{2} + \Phi(Z_n)\right] + \frac{C_I\sigma_n}{\sqrt{2\pi}} \left. e^{-\frac{1}{2}t^2} \right]_{-\infty}^{-Z_n}$$

$$h = C_I\sigma_n Z_n \left[\frac{1}{2} + \Phi(Z_n)\right] + C_I\sigma_n\varphi(Z_n)$$

In an entirely similar manner the expression for $k(P_n, D_n)$ is derived to be:

$$k = -C_S\sigma_n Z_n \left[\frac{1}{2} - \Phi(Z_n)\right] + C_S\sigma_n\varphi(Z_n)$$

$$g = h + k$$

$$= C_I\sigma_n Z_n \left[\frac{1}{2} + \Phi(Z_n)\right] + C_I\sigma_n\varphi(Z_n)$$

$$- C_S\sigma_n Z_n \left[\frac{1}{2} - \Phi(Z_n)\right] + C_S\sigma_n\varphi(Z_n)$$

(13) $$g = \sigma_n Z_n \left[C_I\left(\frac{1}{2} + \Phi(Z)\right) - C_S\left(\frac{1}{2} - \Phi(Z_n)\right)\right]$$

$$+ \sigma_n C_I\varphi(Z_n)\left[C_I + C_S\right]$$

In order to provide a function which is independent of the values of the distribution parameters and of the costs of shortage and inventory define: $r = \frac{C_S}{C_I}$ and divide both sides of (13) by $\sigma_n C_I$

(14) $$\frac{g}{\sigma_n C_I} = Z_n \left[\left(\frac{1}{2} + \Phi(Z)\right) - r\left(\frac{1}{2} - \Phi(Z)\right)\right] + \varphi(Z)\left[1 + r\right]$$

$$= v(Z, r)$$

The function $v(Z,r)$, which is an expression for the non-linear terms in the cost function, is shown plotted in Figure 1. The function is plotted as a family of curves for constant values of r, the ratio of shortage costs to inventory carrying costs. A table of values for the

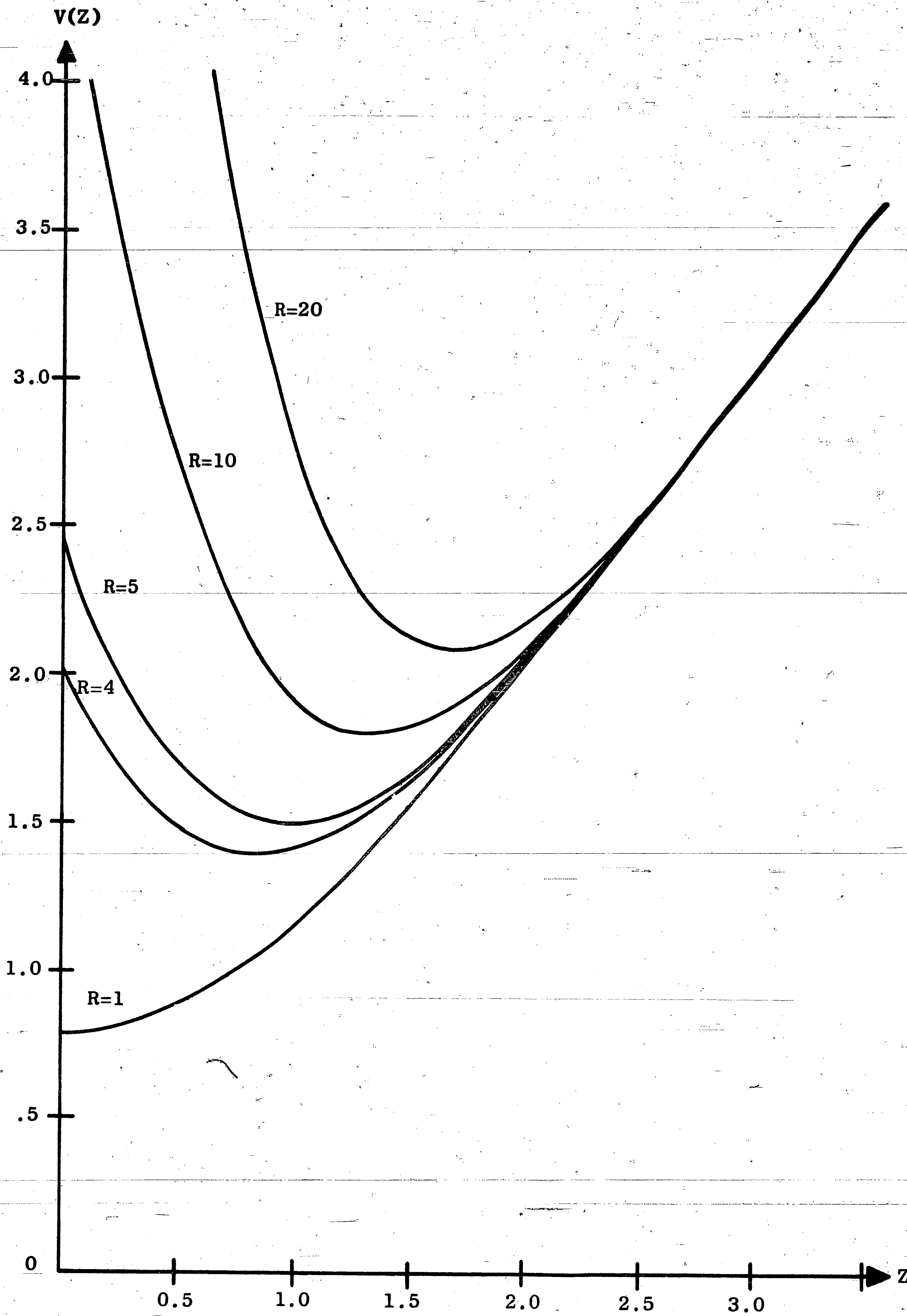

FIGURE 1 NORMALIZED COST FUNCTION

function is given in Appendix I.

We now develop some important properties of the function $v(Z,r)$.

Equation (10) is of the form:

$$g(P) = C_I \int_{-\infty}^{P} (P-D)\ f\ (D)\ dD + C_S \int_{P}^{\infty} (D-P)\ f\ (D)\ dD$$

According to Leibnetz's Rule - Given a Function:

$$g(x) = \int_{h(x)}^{K(x)} f\ (x,y)\ dy$$

$$g'(x) = \frac{dg(x)}{dx} \quad \text{is given by:}$$

$$g'(x) = \int_{h(x)}^{K(x)} \frac{\partial(x,y)}{\partial x}\ dy + f\ [x,K(x)]\ \frac{dK(x)}{dx} - f[x,h(x)]\frac{dh(x)}{dx}$$

Thus:

$$g'(P) = C_I \int_{-\infty}^{P} f\ (d)\ dD - C_S \int_{P}^{\infty} f\ (D)\ dD$$

Using (11) and (12):

$$(15)\quad g'(P) = C_I\ [\tfrac{1}{2} + \Phi(Z)] - C_S\ [\tfrac{1}{2} - \Phi(Z)]$$

$$(16)\quad g''(P) = C_I\ f\ (P) + C_S\ f\ (P)$$

Since $f(P) = \frac{1}{\sqrt{2\pi\sigma}}\ e^{\frac{-1}{2}\left(\frac{P-\mu}{\sigma}\right)^2}$ is positive for all P and C_S and C_I are positive cost coefficients, $g''(P)$ is positive. Thus $g(P)$ has a minimum where $g'(P) = 0$. From (15) $C_I\,[\frac{1}{2} + \Phi(Z)] - C_S\,[\frac{1}{2} - \Phi(Z)] = 0$. Since r is defined as $\frac{C_S}{C_I}$, the function g(P) is minimum where

$\Phi(Z) = \frac{r}{r+1}$ and $Z = \Phi^{-1}\left(\frac{r}{r+1}\right)$

<u>Definition of a Convex Function</u>: A function h(x) is said to be convex in an interval (a,b) if $h''(x) \geq 0$ for all $a \leq x \leq b$.

It follows from this definition and equation (16) that g(P) is convex for all P. This is an important result which will be used in later development.

Thus the production smoothing problem with normally distributed demands in each period of the planning horizon becomes one to:

$$\text{Minimize:}\quad C_r \sum_{n=1}^{N} X_n + C_0 \sum_{n=1}^{N} Y_n + C_h \sum_{n=1}^{N} U_n + C_f \sum_{n=1}^{N} V_n + \sum_{n=1}^{N} V_n + \sum_{n=1}^{N} g_n(P_n)$$

Subject to the constraints:

$$X_n \leq K_n\,[W_0 + \sum_{i=1}^{n} (U_i - V_i)]$$

$$Y_n \leq K_n'\,[W_0 + \sum_{i=1}^{n} (U_i - V_i)]$$

$$P_n \geq \hat{D}_n$$

The function $g_n(P_n)$ has been shown to be convex and since the sum of convex functions is also convex the total objective function is convex.

II-D. An Equivalent Linear Program

A method commonly used to solve this type of problem is to fit a series of straight line segments to the non-linear function, and solve the resultant linear problem. The curves of Figure 1 can be approximated to any desired accuracy by a broken line fit. Obviously as the accuracy required is increased the number of segments required increases with a corresponding increase in the size of the problem. A compromise between accuracy and computational feasibility must be made.

There are two distinct advantages derived from maintaining a linear problem format: (1) Developments made in conjunction with linear programming, such as the notion of duality and sensitivity and shadow price analysis which provide information to evaluate alternative policies without solving the problem for each alternative, can be applied. (2) The availability of computer programs to solve the linear problem is a second advantage. Most computers have a program for the simplex algorithm available as part of their program library, whereas some of the more general non-linear programming algorithms are only available for the very large data-processing systems.

Several characteristics of the function (Figure 1) are useful in minimizing the number of segments required for a suitable fit. Note that $v(Z)$ approaches the line $v(Z) = Z$ asomptotically. Thus, for realistic values r, the portion of the curve beyond $Z = 2.5$ need not be considered. Also implicit in the development is that we are not concerned with negative values for Z. This would imply a probability of less than one-half of satisfying demand which in most industrial situations is an impractical policy.

Since the general shape of the curves is uneffected by the values of the parameters of the distribution, the same broken line approximation will apply for any normal density function and for each of the periods of the planning horizon thus greatly reducing the computational effort required in applying the model.

In Figure 2 below the slopes of the straight line segments are denoted by m_i and the width of the intervals by w_i.

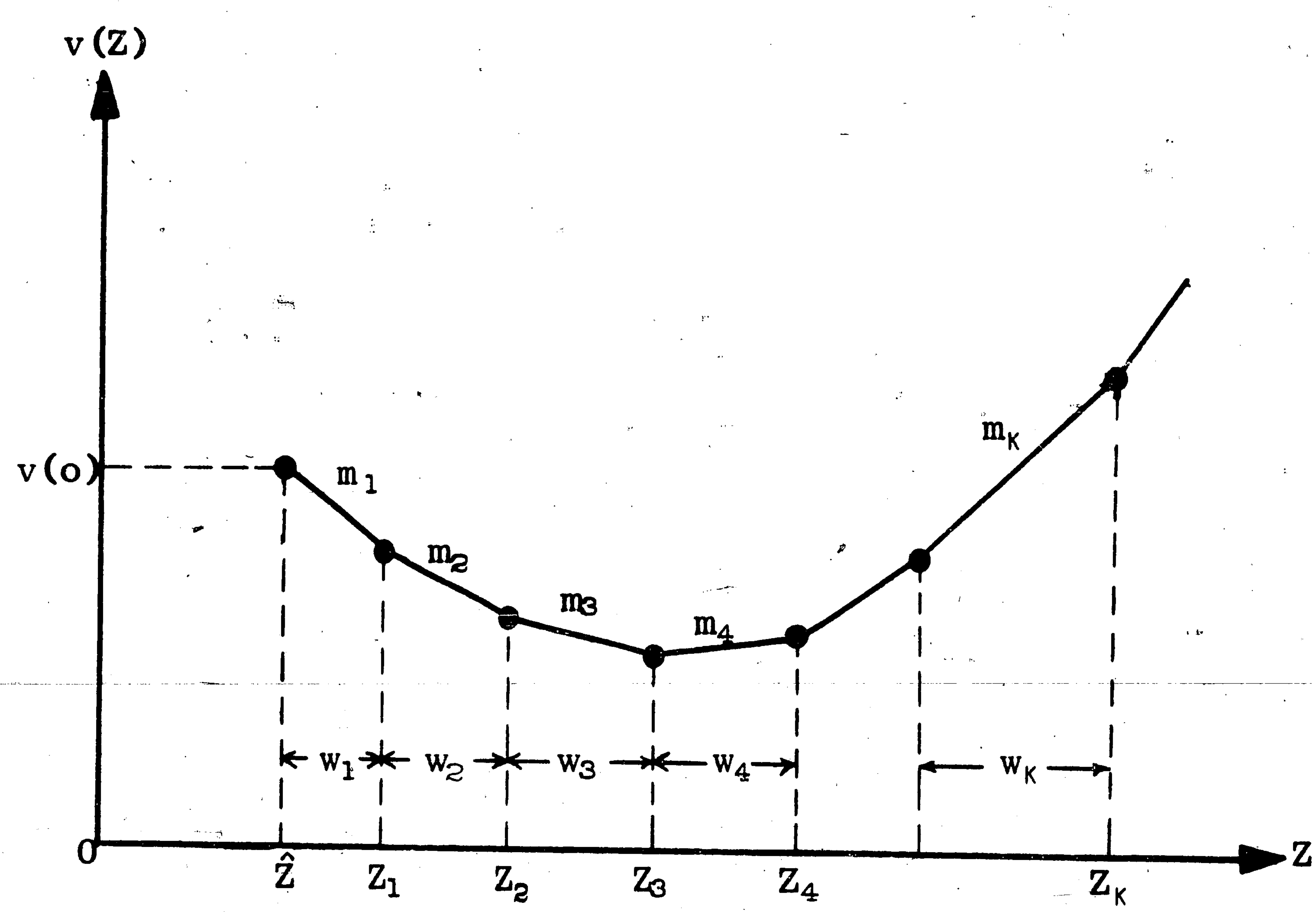

FIGURE 2

The function v(Z) can be approximated in the interval $(\hat{Z}, Z_K)$ by:

$$v(Z) = v_0 + m_1 \Delta Z_1 + \ldots + m_K \Delta Z_K$$

or

$$v(Z) = v_o + \sum_{i=1}^{K} m_i \Delta Z_i$$

Where $\Delta Z_i = Z_i - Z_{i-1}$ $\quad 0 \le \Delta Z_i \le w_i$

And $Z = \sum_{i=1}^{K} \Delta Z_i$

From Equation (14) $g(Z) = \sigma C_I v(Z)$

And from (11) $\quad Z = \frac{P-\mu}{\sigma}$

Therefore $\Delta Z_i = \frac{P_i-\mu}{\sigma} - \frac{P_{i-1}-\mu}{\sigma} = \frac{P_i-P_{i-1}}{\sigma}$

Or, $\quad \Delta Z_i = \frac{\Delta P_i}{\sigma}$

Where $\Delta P_i = P_i - P_{i-1}$

Thus $\quad g = \sigma C_I \left[v_0 + \sum_{i=1}^{K} m_i \left(\frac{\Delta P_i}{\sigma} \right) \right]$

Or,

(17) $\quad g = \sigma C_I v_0 + C_I \sum_{i=1}^{K} m_i \Delta P_i$

Equation (17) is valid for any period n in the planning horizon i.e.,

$$g_n(P_n) = \sigma_n C_I v_0 + C_I \sum_{i=1}^{K} m_i \Delta P_{in}$$

And,

$$\sum_{n=1}^{N} g_n(P_n) = \sum_{n=1}^{N} \sigma_n C_I v_0 + C_I \sum_{n=1}^{N} \sum_{i=1}^{K} m_i \Delta P_{in}$$

$$\sum_{n=1} g_n(P_n) = C + C_I \sum_{n=1}^{N} \sum_{i=1}^{K} m_i \Delta P_{in}$$

Where $C = \sum_{n=1}^{N} \sigma_n C_I v_0$, a constant

Thus the non-linear terms of the cost equations have been reduced to a summation of linear terms plus a constant. The constraints and the objective function are now linear, but the new variables ΔP_{in} must appear in the constraints before the simplex algorithm can be applied. The problem must be set up so as to allow the algorithm to "progress" from $\hat{Z}_n$ (see Figure 2) along v(Z) seeking a minimum for the total cost function. From Figure 2 it is clear that:

$$Z_n = \hat{Z}_n + \sum_{i=1}^{K} \Delta Z_{in}$$

and since $Z_n = \dfrac{P_n - \mu}{\sigma_n}$ and $\hat{Z}_n = \dfrac{\hat{D}_n - \mu_n}{\sigma_n}$

$$P_n = \hat{D}_n + \sum_{i=1}^{K} \Delta P_{in}$$

The entire problem can now be expressed in canonical form for solution by the simplex algorithm.

Minimize: $$C_r \sum_{n=1}^{N} X_n + C_0 \sum_{n=1}^{N} Y_n + C_h \sum_{n=1}^{N} U_n + C_f \sum_{n=1}^{N} V_n + C_e \sum_{n=1}^{N} S_{in}$$

$$+ \sum_{n=1}^{N} \sum_{i=1}^{K} m_i \Delta P_{in} + C$$

Subject to:

$$X_n - \sum_{i=1}^{n} U_i + \sum_{i=1}^{n} v_i + S_{in} = K_n w_0 \qquad n = 1, 2, \ldots, N$$

$$(18) \qquad Y_n - \sum_{i=1}^{n} U_i + \sum_{i=1}^{n} v_i + S_{zn} = K_n' w_0$$

$$\sum_{i=1}^{n} X_i + \sum_{i=1}^{n} y_i + \sum_{i=1}^{K} \Delta P_{in} + A_n = \hat{D}_n - I_0$$

$$\Delta P_{in} + S_{3n} = \sigma_n w_1 \quad i = 1, \ldots, K$$

Where: S_{in} is the unused regular time capacity (idle time) in period n.

S_{2n} is the unused overtime capacity in period n.

A_N is an artificial variable costed sufficiently high so as not to enter the solution.

C_e is the cost of idle time in dollars per hour.

We have shown previously that the objective function is convex. From the convexity of the function it follows that:

$$m_1 \leq m_2 \leq \ldots \leq m_k$$

The slopes (m_i) of the line segments appear as cost coefficients of the variables ΔP_i in the objective function. Since ΔP_{i-1} is costed lower than ΔP_i, (i.e., $m_{i-1} \leq m_i$) any optimizing procedures will exhaust all the variables ΔP_{i-1} before introducing ΔP_i into the solution. Since the cost coefficients assure that the cost curves will be properly generated, we need only constrain the upper bounds of the variables ΔP_i. This provides for a considerable reduction in the size of the problem.

The model as formulated consists of N(4 + K) equations and N(4 + K) original non-basic variables. The factor K is the number of straight line segments required to approximate the cost function.

Dantzig [11] refers to a system of the type given in (18) as a capacitated system since the variables ΔP_{in} are bounded i.e., $0 \leq \Delta P_{in} \leq \sigma w_i$. Since the simplex algorithm restricts the values the variables may assume to positive values, we need not be concerned with the lower bound. However, in order to fix the upper bound one inequation for each variable must be included in the constraint set. Dantzig [11] presents a method by which the capacitated system may be solved by applying the simplex algorithm to the format of the original uncapacitated system, with very simple modifications to insure that the values assigned a variable remain in the range between its upper and lower bounds. This technique will reduce the production smoothing problem to a system of 4N equations with N(4 + K) variables.

III. Conclusions

In order to evaluate the stochastic model a comparison is made between this model and the deterministic model with the random cumulative demand in each period of the planning horizon replaced by the expected value of the demand. Clearly the probability of a stockout with the expected value model is one-half. Since the penalty cost of a stockout is nearly always greater than the cost of inventory the usual procedure is to maintain a buffer or safety stock. In order to compare the two models, the expected costs of carrying inventory and expected penalty costs for shortages must be added to actual production costs which are specified by the expected value (deterministic) solution to obtain the total expected cost of the production program.

With the stochastic model the probability with which demand is to be satisfied is specified by assigning values to the elements of the confidence vector. Here the model seeks a minimum of the total expected cost function directly. It is easily seen that the expected value solution plus a safety stock sufficient to meet demand at the desired probability level is a feasible solution to the stochastic model. Thus, the expected value method with an appropriate safety stock provides a valid though not necessarily optimal solution to the problem, whereas the stochastic model always selects the optimum from the set of feasible solutions. Therefore the stochastic model will always yield a production plan with a total expected less than or equal to that provided by solving the deterministic model in which random demand is replaced by its expected value.

The magnitude of the cost difference in the alternative production schedules is dependent upon the volatility of the mean demand, the variance of the demand density function, the values assigned to the elements of α, the relative magnitudes of the cost parameters, and the starting conditions, i.e., starting workforce and initial inventory.

IV. Recommendations for Further Study

In their work on the chance-constrained formulation, Charnes and Cooper introduce a decision rule of the form $X = Db$ into the problem, where D is an $n \times m$ matrix. If a decision rule of this form is employed in the production smoothing model, the actual observed consumer demand in periods one through $i-1$ can be used when planning production for the i^{th} period.

It has been shown that for certain classes of decision rules the formulation results in a "deterministic equivalent", i.e., a problem not involving any random variables which when solved will provide optimal decision rules. In particular it has been demonstrated that, if the decision rule is linear and the distributions of the random elements are normal, a deterministic equivalent can be obtained [4]. The solution to the equivalent problem is a set of values for the elements of D, the d_{ij}, which when applied as weighting factors to past demand will yield an optimal deterministic production policy for the current planning period.

The development of a production smoothing model using linear decision rules, and a comparison with the model presented in the paper would be of interest. The resultant formulation will likely result in a set of constraints which are at least quadratic and a non-linear objective function.

Most of the modern literature on the subject of stochastic programming is concerned with statistical variation in the b vector.

The case of random cost coefficients has also been treated and can be shown to reduce to an expected value solution if b and c are uncorrelated [11]. Consideration of random elements in the A matrix represents a more difficult task. Tintner [23] presents a solution to a problem of this type, but his method is not computationally feasible for large problems. Although not specifically applicable to the production smoothing problem, the development of a technique to deal with random elements in the A matrix would be a significant contribution to stochastic programming.

Appendix I

Normalized Table of Values

For the Expected Cost Function

NORMALIZED TABLES FOR V(Z)

R / Z	1	2	3	4	5
.0	.7978	1.1968	1.5957	1.9947	2.3936
.1	.8018	1.1527	1.5037	1.8546	2.2055
.2	.8137	1.1206	1.4275	1.7344	2.0413
.3	.8335	1.1002	1.3670	1.6338	1.9005
.4	.8608	1.0913	1.3217	1.5521	1.7826
.5	.8956	1.0934	1.2912	1.4890	1.6868
.6	.9373	1.1060	1.2746	1.4433	1.6120
.7	.9857	1.1286	1.2715	1.4143	1.5572
.8	1.0404	1.1606	1.2808	1.4010	1.5212
.9	1.1008	1.2013	1.3017	1.4021	1.5026
1.0	1.1666	1.2499	1.3332	1.4165	1.4998
1.1	1.2372	1.3058	1.3744	1.4430	1.5116
1.2	1.3122	1.3683	1.4244	1.4805	1.5366
1.3	1.3910	1.4365	1.4821	1.5276	1.5731
1.4	1.4733	1.5099	1.5466	1.5833	1.6199
1.5	1.5586	1.5879	1.6172	1.6465	1.6758
1.6	1.6464	1.6697	1.6929	1.7162	1.7394
1.7	1.7365	1.7548	1.7731	1.7914	1.8096
1.8	1.8285	1.8428	1.8571	1.8713	1.8856
1.9	1.9221	1.9331	1.9442	1.9552	1.9663
2.0	2.0169	2.0254	2.0339	2.0424	2.0509
2.1	2.1129	2.1194	2.1258	2.1323	2.1388
2.2	2.2097	2.2146	2.2195	2.2244	2.2293
2.3	2.3073	2.3110	2.3146	2.3183	2.3220
2.4	2.4054	2.4081	2.4108	2.4135	2.4162
2.5	2.5040	2.5060	2.5080	2.5100	2.5120
2.6	2.6029	2.6043	2.6058	2.6073	2.6087
2.7	2.7021	2.7031	2.7042	2.7052	2.7063
2.8	2.8015	2.8022	2.8030	2.8037	2.8045
2.9	2.9010	2.9015	2.9021	2.9026	2.9031
3.0	3.0007	3.0011	3.0015	3.0019	3.0022
3.1	3.1005	3.1007	3.1010	3.1013	3.1015
3.2	3.2003	3.2005	3.2006	3.2008	3.2010
3.3	3.3002	3.3004	3.3005	3.3006	3.3008
3.4	3.4001	3.4002	3.4002	3.4003	3.4004
3.5	3.5001	3.5001	3.5002	3.5003	3.5003
3.6	3.6000	3.6001	3.6001	3.6001	3.6002
3.7	3.7000	3.7000	3.7000	3.7000	3.7000
3.8	3.8000	3.8000	3.8000	3.8001	3.8001
3.9	3.9000	3.9000	3.9000	3.9000	3.9000

NORMALIZED TABLES FOR V(Z)

R / Z	6	7	8	9	10
.0	2.7925	3.1915	3.5904	3.9894	4.3883
.1	2.5565	2.9074	3.2583	3.6093	3.9602
.2	2.3482	2.6551	2.9620	3.2689	3.5758
.3	2.1673	2.4341	2.7008	2.9676	3.2343
.4	2.0130	2.2435	2.4739	2.7043	2.9348
.5	1.8846	2.0824	2.2802	2.4780	2.6758
.6	1.7806	1.9493	2.1180	2.2867	2.4553
.7	1.7001	1.8430	1.9859	2.1287	2.2716
.8	1.6414	1.7616	1.8818	2.0020	2.1222
.9	1.6030	1.7034	1.8039	1.9043	2.0047
1.0	1.5831	1.6664	1.7497	1.8331	1.9164
1.1	1.5802	1.6489	1.7175	1.7861	1.8547
1.2	1.5927	1.6488	1.7049	1.7610	1.8171
1.3	1.6187	1.6642	1.7097	1.7553	1.8008
1.4	1.6566	1.6933	1.7299	1.7666	1.8033
1.5	1.7051	1.7344	1.7637	1.7930	1.8223
1.6	1.7626	1.7859	1.8091	1.8324	1.8556
1.7	1.8279	1.8462	1.8645	1.8828	1.9010
1.8	1.8999	1.9142	1.9284	1.9427	1.9570
1.9	1.9773	1.9884	1.9994	2.0105	2.0215
2.0	2.0594	2.0679	2.0764	2.0849	2.0933
2.1	2.1453	2.1517	2.1582	2.1647	2.1712
2.2	2.2342	2.2391	2.2440	2.2489	2.2537
2.3	2.3257	2.3293	2.3330	2.3367	2.3404
2.4	2.4189	2.4216	2.4243	2.4271	2.4298
2.5	2.5140	2.5160	2.5180	2.5200	2.5220
2.6	2.6102	2.6117	2.6131	2.6146	2.6161
2.7	2.7073	2.7084	2.7094	2.7105	2.7115
2.8	2.8052	2.8060	2.8067	2.8075	2.8082
2.9	2.9036	2.9042	2.9047	2.9052	2.9057
3.0	3.0026	3.0030	3.0034	3.0038	3.0041
3.1	3.1018	3.1021	3.1023	3.1026	3.1028
3.2	3.2012	3.2013	3.2015	3.2017	3.2018
3.3	3.3009	3.3010	3.3012	3.3013	3.3014
3.4	3.4005	3.4005	3.4006	3.4007	3.4008
3.5	3.5004	3.5005	3.5005	3.5006	3.5007
3.6	3.6002	3.6002	3.6003	3.6003	3.6003
3.7	3.7000	3.7001	3.7001	3.7001	3.7001
3.8	3.8001	3.8001	3.8002	3.8002	3.8002
3.9	3.9000	3.9000	3.9000	3.9000	3.9000

NORMALIZED TABLES FOR V(Z)

R / Z	11	12	13	14	15
.0	4.7872	5.1862	5.5851	5.9841	6.3830
.1	4.3111	4.6621	5.0130	5.3639	5.7149
.2	3.8827	4.1895	4.4964	4.8033	5.1102
.3	3.5011	3.7679	4.0346	4.3014	4.5682
.4	3.1652	3.3956	3.6261	3.8565	4.0870
.5	2.8736	3.0714	3.2692	3.4670	3.6648
.6	2.6240	2.7927	2.9613	3.1300	3.2987
.7	2.4145	2.5574	2.7002	2.8431	2.9860
.8	2.2424	2.3626	2.4828	2.6030	2.7232
.9	2.1052	2.2056	2.3061	2.4065	2.5069
1.0	1.9997	2.0830	2.1663	2.2496	2.3329
1.1	1.9233	1.9919	2.0605	2.1291	2.1978
1.2	1.8732	1.9293	1.9854	2.0415	2.0976
1.3	1.8463	1.8918	1.9374	1.9829	2.0284
1.4	1.8399	1.8766	1.9133	1.9499	1.9866
1.5	1.8516	1.8809	1.9102	1.9395	1.9688
1.6	1.8788	1.9021	1.9253	1.9486	1.9718
1.7	1.9193	1.9376	1.9559	1.9742	1.9924
1.8	1.9713	1.9855	1.9998	2.0141	2.0284
1.9	2.0326	2.0436	2.0547	2.0657	2.0768
2.0	2.1018	2.1103	2.1188	2.1273	2.1358
2.1	2.1776	2.1841	2.1906	2.1971	2.2035
2.2	2.2586	2.2635	2.2684	2.2733	2.2782
2.3	2.3440	2.3477	2.3514	2.3551	2.3587
2.4	2.4325	2.4352	2.4379	2.4406	2.4433
2.5	2.5240	2.5260	2.5280	2.5300	2.5320
2.6	2.6175	2.6190	2.6204	2.6219	2.6234
2.7	2.7126	2.7136	2.7147	2.7157	2.7168
2.8	2.8090	2.8097	2.8105	2.8112	2.8120
2.9	2.9063	2.9068	2.9073	2.9079	2.9084
3.0	3.0045	3.0049	3.0053	3.0057	3.0060
3.1	3.1031	3.1034	3.1036	3.1039	3.1042
3.2	3.2020	3.2022	3.2024	3.2025	3.2027
3.3	3.3016	3.3017	3.3019	3.3020	3.3021
3.4	3.4008	3.4009	3.4010	3.4011	3.4011
3.5	3.5007	3.5008	3.5009	3.5009	3.5010
3.6	3.6004	3.6004	3.6004	3.6005	3.6005
3.7	3.7001	3.7001	3.7001	3.7001	3.7002
3.8	3.8002	3.8003	3.8003	3.8003	3.8003
3.9	3.9000	3.9000	3.9000	3.9000	3.9000

NORMALIZED TABLES FOR V(Z)

R / Z	16	17	18	19	20
.0	6.7819	7.1809	7.5798	7.9788	8.3777
.1	6.0658	6.4167	6.7677	7.1186	7.4695
.2	5.4171	5.7240	6.0309	6.3378	6.6447
.3	4.8349	5.1017	5.3684	5.6352	5.9020
.4	4.3174	4.5478	4.7783	5.0087	5.2391
.5	3.8626	4.0604	4.2582	4.4560	4.6538
.6	3.4673	3.6360	3.8047	3.9734	4.1420
.7	3.1289	3.2718	3.4146	3.5575	3.7004
.8	2.8434	2.9636	3.0838	3.2040	3.3242
.9	2.6074	2.7078	2.8082	2.9087	3.0091
1.0	2.4162	2.4995	2.5828	2.6662	2.7495
1.1	2.2664	2.3350	2.4036	2.4722	2.5408
1.2	2.1538	2.2099	2.2660	2.3221	2.3782
1.3	2.0740	2.1195	2.1650	2.2106	2.2561
1.4	2.0233	2.0599	2.0966	2.1333	2.1699
1.5	1.9981	2.0274	2.0567	2.0861	2.1154
1.6	1.9950	2.0183	2.0415	2.0648	2.0880
1.7	2.0107	2.0290	2.0473	2.0656	2.0839
1.8	2.0426	2.0569	2.0712	2.0855	2.0997
1.9	2.0878	2.0989	2.1099	2.1210	2.1320
2.0	2.1443	2.1528	2.1613	2.1698	2.1782
2.1	2.2100	2.2165	2.2230	2.2294	2.2359
2.2	2.2831	2.2880	2.2929	2.2978	2.3026
2.3	2.3624	2.3661	2.3698	2.3734	2.3771
2.4	2.4460	2.4487	2.4514	2.4542	2.4569
2.5	2.5340	2.5360	2.5380	2.5401	2.5421
2.6	2.6248	2.6263	2.6278	2.6292	2.6307
2.7	2.7178	2.7189	2.7199	2.7210	2.7220
2.8	2.8127	2.8135	2.8142	2.8150	2.8157
2.9	2.9089	2.9094	2.9100	2.9105	2.9110
3.0	3.0064	3.0068	3.0072	3.0076	3.0079
3.1	3.1044	3.1047	3.1049	3.1052	3.1055
3.2	3.2029	3.2030	3.2032	3.2034	3.2036
3.3	3.3023	3.3024	3.3025	3.3027	3.3028
3.4	3.4012	3.4013	3.4014	3.4014	3.4015
3.5	3.5011	3.5011	3.5012	3.5013	3.5013
3.6	3.6005	3.6006	3.6006	3.6006	3.6007
3.7	3.7002	3.7002	3.7002	3.7002	3.7002
3.8	3.8004	3.8004	3.8004	3.8004	3.8005
3.9	3.9000	3.9000	3.9000	3.9001	3.9001

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Professional Organizations

American Institute of Electrical and Electronic Engineers

Registered Professional Engineer - State of Ohio